Particle Transport in the Magnetosphere: A New Diffusion Model

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Abstract

Previous transport models of the plasmasphere, ring current and radiation belts have considered either diffusive or convective transport, but not both. Since the ∇B drift speed is proportional to energy, analyses of particles with E>100 keV have generally used only ∇B drift and diffusive transport. Conversely, particles of E<30 keV have been treated with only E×B drift and convective transport. The ring current, which lies between these two regimes, cannot be described by either mechanism alone, so that these standard models, though widely used and otherwise applicable, fail in detail to describe the trapped ion population as observed with modern instrumentation. We develop a formalism that can describe both diffusive and convective transport in a completely general way, including UT, LT, radial, and pitch angle dependence, with arbitrary magnetic and electric field models. The formalism is general enough to rederive the electric or magnetic diffusion coefficients for trapped particles without the simplifying assumption of axial symmetry or a vanishing convection electric field, thus improving on the standard coefficients derived nearly 30 years ago. More importantly, we include the previously neglected diffusion term due to a localized, non-global perturbation of the fields, and show that under some circumstances this term may dominate over the resonant diffusion caused by global perturbations.

Introduction and Data Example

Two types of models have been used to describe the ring current (1<E<300 keV), diffusion models [e.g., *Sheldon* 1994a] and convection models [e.g., *Fok et al.* 1995]. Both models are very sophisticated yet have deviations of an order of magnitude from the data, showing that neither approach alone can describe the ring current. A variant on the second approach used MonteCarlo simulations of convective transport but with the impulsive bursty characteristics of the storm-time electric field [*Chen et al.*, 1994], and found that quasi-diffusive transport, faster than drift-resonant [*Schulz and Lanzerotti*, 1974] diffusive transport was needed in such simulations to describe the ring current ions. Therefore, not only do we need a formalism for including diffusion in a convection model, but we need a robust definition of diffusion as well.

In addition to simulations, there is compelling empirical evidence that no present convection or diffusion model can adequately explain the data. One example from the AMPTE mission will suffice. On days 138-143 of 1987, the CHEM instrument observed the quietest ring current in its four-year mission, as determined by both ground based indices and *in situ* measured densities. A plot of those six radial passes (see Figure 1) shows a distinct peak around L=2.8 that surfaced as the magnetosphere quieted (Days 139-142), but appears diffused during more active periods (Day 138). Now this peak is not the result of a time-dependent transport due to electric fields or currents, since the magnetosphere was in an extremely quiescent state, but arises from the convective access of plasmasheet ions to the inner ring current, which can be observed only in a realistic convection model [*Sheldon*, 1994b]. Although one might include such *ad hoc* source terms in either the diffusive or the convective models above, the data show other discrepancies not so easily included. Figure 1

Time-dependent convection models can only move particles from the plasmasheet on open drift orbits, and use time-dependence to get particles onto closed, ring current drift orbits. In any quasi-static time-independent magnetosphere, however, the particles on closed orbits would decay with time, producing sharp boundaries between open and closed drift orbits. Thus the ring current should be essentially empty for any protracted period of magnetospheric quieting. Since the loss times typically quoted for the ring current are about 10 hours, the five days of quiet shown here should have produced an empty ring current out to about L=6. Clearly an additional transport mechanism is operating in our data.

The standard diffusion model implements such a quiet-time radial transport, but typically with a steep power law dependence on L-shell. So even if we include a low-L source to mimic the convective access, there would not be the enhanced diffusion observed around L=4. That is, the diffusion models predict a steep flux gradient at L∼5 when the diffusion rate drops below the charge-exchange loss rate. Now the flattening of the flux gradient at L=4 on day 138 can only occur if the diffusion rate is again greater than the loss rate. Since the loss rate increases monotonically as one approaches the atmospheric source of neutral hydrogen, this implies that the diffusion rate show a local maxima at L=4. But the standard diffusion models all have diffusion coefficients which monotonically decrease with L in disagreement with the data [*West et al.*, 1981, *Sheldon and Hamilton*, 1993].

As before, we can again add a second internal source of diffusion to account for the non-monotonic shape at L=4 [*Sheldon*, 1994a], which does not change the character of the solution at L$>$5. Yet even with this highly *ad hoc* model tuned to explain L$<$5 fluxes, we cannot explain the change in the slope of the phase space density seen around L=5 for days 138–140 to days 141–142. Since the slope depends not on the amplitude of the diffusion coefficient but its power law exponent, this change in slope implies a temporal change in the power law exponent of the electric diffusion coefficient. This is only possible in current diffusion theory if the spectral index of the perturbations of the solar wind driver should change in exactly the same manner. In this case, we would require a solar wind perturbation spectral index change from f^{-2} to about f^{-6}, which appears rather unsupported by solar wind observations. Thus it appears difficult to account for temporal changes in the ring current using current diffusion models whenever long time averages, which tend to introduce substantial numerical diffusion, are not performed.

We propose that the above theoretical and observational problems can be solved, first by transforming the convection problem into more intuitive coordinates and then considering the diffusion of these convection drift orbits using recent descriptions of the diffusion process. In Section 2 we describe the coordinate transformation that enables us to separate convection from diffusion. In Section 3 we develop a standard approach to diffusion in energy-space and show how we can derive diffusion coefficients for random perturbations in the fields. In Section 4 we describe the differences between resonant diffusion and our derivation, and when this additional diffusion is important.

Hamiltonian Convection

Several approaches have been used to solve the problem of ion trajectories in the earth's magnetosphere. Most rely on integrating the forces along a particle trajectory, a Lagrangian approach. Our approach uses conservation of energy and the first two adiabatic invariants along a bounce-averaged trajectory so that the orbit is determined by the isoenergy contours for a prescribed electric and magnetic field model. This Hamiltonian approach has the advantage that errors do not propagate and that trajectories are found quickly and accurately. The relevant equation is given by *Whipple* [1978],

$$H_0 = K.E. + P.E. = \mu B_m + qU \quad (1)$$

where H_0 is the unperturbed Hamiltonian consisting of the sum of kinetic (K.E.) and potential (P.E.) energies. At the mirror point, all the K.E. is in the perpendicular motion, so that the magnetic moment, μ, multiplied by the mirror point magnetic field strength, B_m, gives the ion K.E. The potential energy is assumed to be well described by a scalar field, U, times the charge, q.

The Hamiltonian method outlined above is completely general for all pitch angles. The electrostatic potential U is assumed to be constant along a field line, and therefore is independent of pitch angle. However B_m is clearly pitch angle dependent, and must be known for every value of the second adiabatic invariant J or K [*Sheldon and Gaffey*, 1993].

Birmingham [1984] discusses the conditions under which the first two invariants are not conserved, which for a "realistic" model field, e.g. Tsyganenko 89, occurs near midnight at L=8–10 for 100 keV particles (private communication, B. Anderson 1996). Time dependence is also possible if changes in B_m and U are "slow enough", or quasi-static, a property met by all models previously described. Thus the boundary conditions under which the above formulation is applicable are the ionosphere, the magnetopause, and the energy-dependent "Birmingham boundary" around 10 R_E at midnight.

In the adiabatic regime of a quasi-static magnetosphere, ions neither gain nor lose total energy so that we can write the derivative,

$$0 = \frac{dH_0}{dt} = \mu \frac{\partial B_m}{\partial t} + q \frac{\partial U}{\partial t} \Rightarrow \frac{\partial U}{\partial B_m} = -\frac{\mu}{q} \quad (2)$$

i.e., if we convert from real (x, y)-space into (U, B_m)-space, then the ion trajectories are straight lines with the slope $-\mu/q$. Not only does this greatly enhance our intuition for magnetospheric convection, but it clearly separates the competing transport mechanisms of diffusion and convection. Although diffusion is occurring in all directions, it is only apparent when no other transport mechanism, such as convection, overwhelms its small displacement. Thus we can identify convection with the motion along a drift orbit line, and diffusion as the motion transverse to drift lines.

This definition of diffusion overcomes most of the difficulties encountered in the diffusion models discussed above. They all assume that the drift orbits are circular after removing the effect of a non-axisymmetric magnetic field so that diffusion then corresponds to radial transport. This is clearly not true for ring current ions, which because of oppositely directed ∇B and $E \times B$ drifts, orbit on distorted ellipses, bananas, and other unnamed figures. Thus we are unable to compute ring current diffusion coefficients from previous theory and instead rederive the diffusion coefficient in a Hamiltonian formalism.

Non-Resonant Diffusion

Let us write a simple extension of the above quasi-steady state Hamiltonian including a small perturbation term,

$$H = H_0 + H_1 = \mu B_m + qU + \mu\delta B_m + q\delta U \qquad (3)$$

We assume that the average value of this Hamiltonian is the steady state, H_0, which is to say that fluctuations in the magnetosphere neither energize nor damp the total energy of the system, e.g. a steady ramp up of the electric field would violate this assumption. Near the drift resonance, $\langle H_1 \rangle \neq 0$, unless we also average over drift phase. This average over drift phase is potentially the thorniest issue in calculating both resonant and non-resonant diffusion, as well as in distinguishing irreversible, entropy-increasing diffusion from reversible "migration", but in the following discussion we assume that a time-average is sufficient.

If we consider an ensemble of nearly coincident orbits, with nearly the same total energy, then we can describe their variance as

$$H_{rms}^2 = \left\langle (H - \langle H \rangle)^2 \right\rangle \cong \left\langle \mu^2 \delta B^2 \right\rangle + \left\langle q^2 \delta U^2 \right\rangle \qquad (4)$$

where we assume that δB and δU are uncorrelated. This is probably not true in general, though we believe that the cross term $\delta B \delta U$ will be smaller than the other terms, which can be seen by noting that changes in U will occur only in the ionosphere, which because of its conductivity will "freeze in" any fluctuation of magnetic foot points.

Now diffusion in energy space is nothing but broadening of the variance that we have just defined [*Reif*, 1965],

$$2D = \frac{d}{dt}(H_{rms}^2) = \frac{d}{dt}\left(\left\langle \mu^2 \delta B^2 \right\rangle + \left\langle q^2 \delta U^2 \right\rangle\right) \qquad (5)$$

This broadening is in the direction perpendicular to the convection velocity since convective velocities are generally an order of magnitude faster than diffusive broadening so that we do not observe the component of diffusion parallel to convection. This direction can be calculated from the slope, $\tan(\theta) = -\mu/q$, where θ is the angle wrt B, the x-axis. The diffusive direction requires that a fluctuation in B be multiplied by $\sin(\theta)$, and conversely that a fluctuation in U be multiplied by $\cos(\theta)$. Thus we have,

$$D = \frac{\sin(\theta)}{2}\frac{d}{dt}\left(\langle \mu \delta B \rangle^2 - \frac{q}{\mu}\langle q \delta U \rangle^2\right) \qquad (6)$$

There are several features of this expression that indicate self-consistency and coherence of our approach. For high energies ($\mu \gg 1$) the expression is dominated by magnetic fluctuations, as is well known for radiation belt ions. For low energies ($\mu \ll 1$) the expression is dominated by electric field fluctuations, as expected for plasmasheet ions. Thus we have spanned the entire range of ring current energies with this one expression.

The diffusion coefficient derived above can be converted from energy space to real space by using spatial derivatives of H for a specified magnetic and electric field configuration. For example, a fluctuating Volland-Stern electric field where the fluctuations occur not in the first, corotation term, but in the second, convection term, can be written,

$$\begin{aligned} U &= \frac{A}{L} + CL^2 \sin(\phi) \qquad (7)\\ \delta U &= L^2 \sin(\phi)\delta C \\ \frac{dH}{dL} &= q\left(-\frac{A}{L^2} + 2CL\sin(\phi)\right) \end{aligned}$$

where ϕ is the LT from midnight, and A, C are the coefficients for corotation and cross-tail field. Then at low energy ($\mu \ll 1$),

$$D_{LL}^E = \frac{\cos(\theta) L^8 \sin^2(\phi)}{2(2CL^3\sin(\phi) - A)^2}\frac{d}{dt}(\delta^2 C) \qquad (8)$$

Thus for $A \ll 2C\sin(\phi)L^3$, $D_{LL}^E \sim L^2$, whereas for $A \gg 2C\sin(\phi)L^3$, $D_{LL}^E \sim L^8$. That is, active Kp periods have a smaller radial gradient for diffusion than quiet periods, which is precisely the observed temporal deviation shown in Figure 1. We can now explain time-dependence in the power-law exponent of the diffusion coefficient.

Non-Global Diffusion

There are significant differences between the standard derivation of the resonant diffusion coefficient and that for our Hamiltonian diffusion coefficient as shown above. The resonant diffusion derivations of *Parker* [1960], *Fälthammar* [1965], and *Nakada and Mead* [1965], assume that the major contribution to the diffusion coefficient is the resonant component. Since the magnitude and direction of the resonant component are phase dependent, changing sign for day and night sides of the magnetosphere, a second assumption is that magnetospheric fluctuations are phase mixed in time so that the average effect is diffusive. This classical approach makes two critical assumptions: the perturbations are incoherent in time and coherent in space. That is, the perturbations are resonant, having a coherence time that is short with respect to the system, and global, causing all drift paths to shift in phase together. But there are also situations in which violation of the second assumption, coherence in space, might lead to a larger diffusion coefficient than the resonant model. That is, under certain circumstances, turbulent spatial diffusion should be more effective than temporal, phase-mixing diffusion.

Arguments were presented nearly 30 years ago in the context of magnetized solar wind plasma, (e.g., *Taylor and McNamara* [1971], *Matthaeus et al.* [1995] and references therein), that resonant, "slab" diffusion depends on the second power of the fluctuations ($D_{LL} \propto (\delta B/B_0)^n$) where n=2, whereas non-resonant "2D" diffusion depends only on the first power, n=1. If this be the case for the magnetosphere, and crucial assumptions have yet to be proven, then in the limit of $\delta B/B_0 \to 0$, (i.e. for quiet magnetospheric conditions) this incoherent, non-resonant diffusion may dominate over the resonant contribution. Since these are exactly the conditions under which the data were taken, it is possible that we are perhaps seeing a time period dominated by non-resonant diffusion. At the very least, we have demonstrated that the superiority of resonant diffusion over non-resonant diffusion cannot be taken for granted, it must be proven. In a future paper, we shall apply this model to the data set described in the introduction.

Conclusions

We have attempted to show how the use of a Hamiltonian formalism can help us to calculate diffusion in the magnetosphere and separate it from convective effects. This means that we can now treat ring current and plasmasheet ions with a single consistent theoretical approach that incorporates both diffusion and convection. We go on to show that this treatment can also improve the standard diffusion coefficients both by generalizing magnetic impulses to any type of electromagnetic perturbation and by incorporating recent insights into the nature of "2D" and "slab" diffusion. We believe that this reformulation of the diffusion paradigm will be of great value to modellers attempting to fit modern data sets. Not only has the basis set of diffusion models been enriched, but convection has been incorporated in a way that permits a fluid description of magnetospheric particles from 1 eV–100 MeV.

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References

Birmingham, T. J. Pitch angle dispersion in the jovian magnetodisk. *J. Geophys. Res.*, *89*, 2699–2710, 1984.

Chen, M. W., L. R. Lyons, and M. Schulz. Simulations of phase space distributions of storm time proton ring current. *J. Geophys. Res.*, *99*, 5745–5759, 1994.

Fälthammar, C.-G. Effects of time-dependent electric fields on geomagnetically trapped radiation. *J. Geophys. Res.*, *70*, 2503–2516, 1965.

Fok, M.-C., T. E. Moore, J. U. Kozyra, G. C. Ho, and D. C. Hamilton. Three-dimensional ring current decay model. *J. Geophys. Res.*, *100*, 9619–9632, 1995.

Matthaeus, W. H., P. C. Gray, D. H. Pontius, Jr, and J. W. Bieber. Spatial structure and field-line diffusion in transverse magnetic turbulence. *Phys. Rev. Lett.*, *75*, 2136–2139, 1995.

Nakada, M. P. and G. D. Mead. Diffusion of protons in the outer radiation belt. *J. Geophys. Res.*, *70*, 4777–4791, 1965.

Parker, E. N. Geomagnetic fluctuations and the form of the outer zone of the van allen radiation belt. *J. Geophys. Res.*, *65*, 3117–3130, 1960.

Reif, F. *Fundamentals of Statistical and Thermal Physics*. McGraw-Hill, New York, 1965.

Schulz, M. and L. J. Lanzerotti. *Particle Diffusion in the Radiation Belts*. Springer-Verlag, New York, 1974.

Sheldon, R. B. and J. D. Gaffey, Jr. Particle tracing in the magnetosphere: New algorithms and results. *Geophys. Res. Lett.*, *20*, 767–770, 1993.

Sheldon, R. B. and D. C. Hamilton. Ion transport and loss in the earth's quiet ring current 1. data and standard model. *J. Geophys. Res.*, *98*, 13,491–13,508, 1993.

Sheldon, R. B. Ion transport and loss in the earth's quiet ring current 2. diffusion and magnetosphere-ionosphere coupling. *J. Geophys. Res.*, *99*, 5705–5720, 1994a.

Sheldon, R. B. Plasmasheet convection into the inner magnetosphere during quiet conditions. In D. N. Baker, editor, *Solar Terrestrial Energy Program: COSPAR Colloquia Series*, volume 5, pages 313–318, New York, 1994b. Pergamom Press.

Taylor, J. B. and B. McNamara. *Phys. Fluids*, *14*, 1492–1499, 1971.

West, Jr, H. I., R. M. Buck, and G. T. Davidson. The dynamics of energetic electrons in the earth's outer radiation belt during 1968 as observed by the lawrence livermore national laboratory's spectrometer on ogo 5. *J. Geophys. Res.*, *86*, 2111–2142, 1981.

Whipple, Jr, E. C. (u, b, k) coordinates: A natural system for studying magnetospheric convection. *J. Geophys. Res.*, *83*, 4318–4326, 1978.

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Figure 1. Phase space density of 10-20 MeV/G H^+ on six successive AMPTE/CHEM passes.